On highly anisotropic big bang singularities

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Question: Are spatially flat and isotropic solutions to Einstein's equations coupled to a radiation fluid stable?

The expansion normalised Weingarten map

Crushing singularity: (M, g), where $M = \bar{M} \times I$; $I = (t_-, t_+)$; $\bar{M}_t := \bar{M} \times \{t\}$ spacelike Cauchy hypersurfaces; mean curvature θ of $\bar{M}_t$ tends to infinity as $t \to t_-+$.

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Isotropy: $\mathcal{K}$ is a multiple of the identity.

Oscillatory and quiescent singularities

$\mathcal{K}$ symmetric w.r.t. $\bar{g}$ (the Riemannian metric induced on $\bar{M}_t$) $\rightsquigarrow$ real eigenvalues ℓ_A, $A = 1, \dots, n$, whose sum equals 1.

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$$\ell_+ := \frac{3}{2}\left(\ell_2 + \ell_3 - \frac{2}{3}\right), \quad \ell_- := \frac{\sqrt{3}}{2}(\ell_2 - \ell_3).$$

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Oscillatory singularity: not quiescent.

The Kasner spacetimes

Kasner solutions to Einstein's vacuum equations:

$$g_{\mathrm{K}} := -dt \otimes dt + \sum_{i=1}^{n} t^{2p_i} dx^i \otimes dx^i$$

on $M_{\mathrm{K}} := \mathbb{R}^n \times (0, \infty)$. Here $\sum_{i=1}^{n} p_i = 1$, $\sum_{i=1}^{n} p_i^2 = 1$.

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Flat Kasner solutions: If one $p_i = 1$, then $p_j = 0$ for $j \neq i \leftrightarrow$ flat Kasner solutions.

The Kasner circle

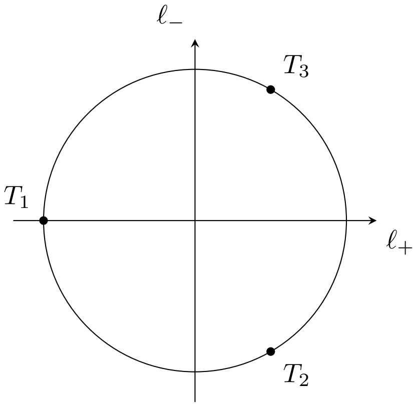


Figure: The Kasner circle with the special points T_i, $i = 1, 2, 3$, indicated.

The Bianchi type I radiation fluid solutions

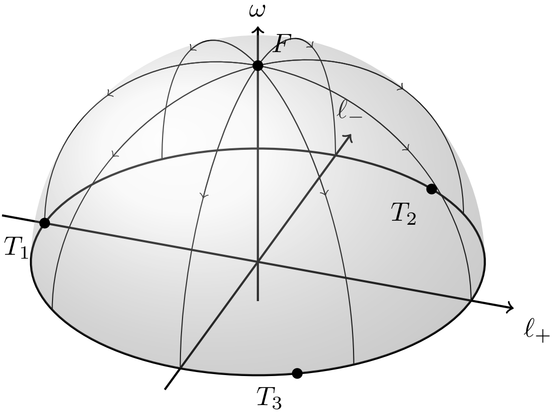
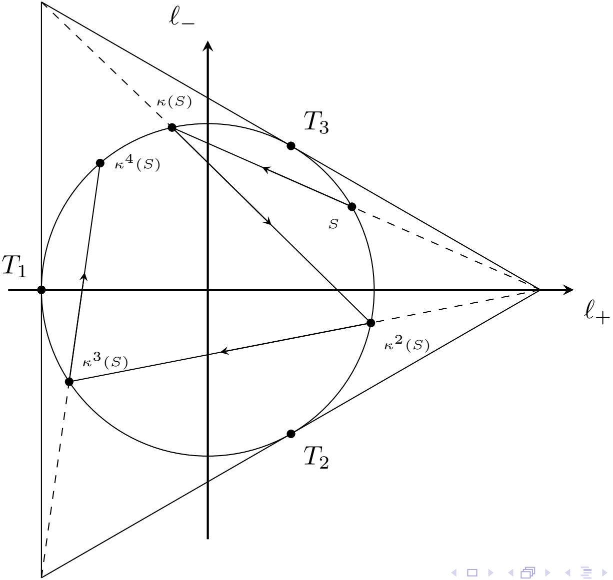


Figure: Dynamics of Bianchi type I radiation fluid solutions. $F \leftrightarrow$ isotropic solutions.

The Kasner map



Bianchi type I stiff fluid solutions

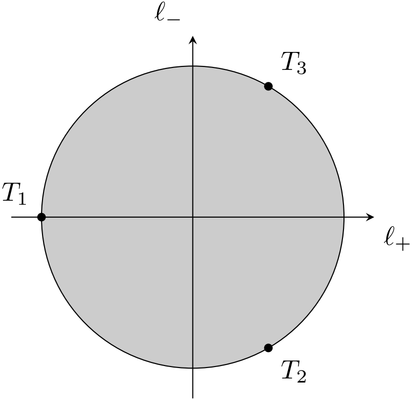


Figure: A projection of the Bianchi type I stiff fluid state space ($\gamma = 2$) to the $\ell_+\ell_-$-plane. The state space consists of fixed points.

Stiff fluid solutions, stable regime

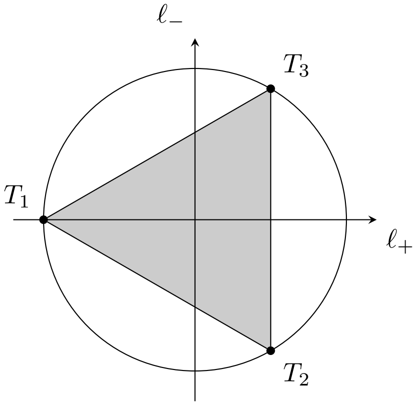


Figure: The gray area indicates the subset in which stable big bang formation is expected in the stiff fluid/scalar field setting.

Mathematical results, big bang singularities

Spatially homogeneous setting:

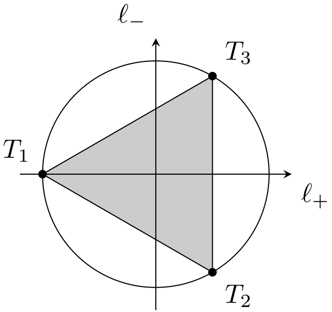
- Lower symmetry types $\rightsquigarrow$ quiescent behaviour.
- Higher symmetry types $\rightsquigarrow$ oscillatory behaviour.
- Stiff fluids suppress oscillations.

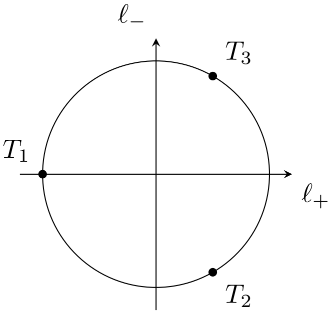
$\mathbb{T}^3$**-Gowdy symmetric setting:** There are typically spikes, inducing significant spatial variations.
Conclusion: Consider the asymptotics in $J^+(\gamma)$.

Specifying data on the singularity

Andersson and Rendall: Can specify real analytic data (without symmetries), pointwise belonging to the triangle.

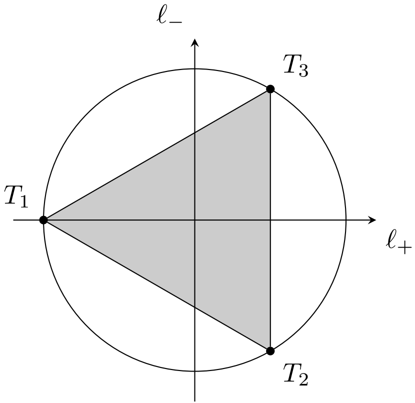
Fournodavlos and Luk: Can specify smooth data (non-generic but without symmetries) on the Kasner circle.





Stable big bang formation

Rodnianski and Speck: Stable big bang formation in a neighbourhood of the origin.



Logarithmic volume density

Volume density: Let $\bar{g}_{\mathrm{ref}}$ be a reference metric on $\bar{M}$. Then the volume density φ is defined by

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Logarithmic volume density: $\varrho := \ln \varphi$.

Assumption: $\varrho \to -\infty$ in the direction of the singularity.

Assumptions, silence and non-degeneracy

Silence: (M, g) has a crushing singularity with $\theta > 0$

- $\hat{g} := \theta^2 g$.
- $\check{K}$ Weingarten map associated with $\hat{g}$.

Assume that there is an $\epsilon_{\mathrm{Sp}} > 0$:

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Non-degeneracy: The eigenvalues of $\mathcal{K}$ are distinct and there is an $\epsilon_{\mathrm{nd}} > 0$ such that the distance between different eigenvalues is bounded from below by ϵ_{nd}.

Frame

Let $\ell_1 < \cdots < \ell_n$ and $\{X_A\}$ be such that

- $\mathcal{K}X_A = \ell_A X_A$ (no summation),
- $\bar{g}_{\mathrm{ref}}(X_A, X_A) = 1$.

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Frame: $\hat{U}$ combined with $\{X_A\}$ is an orthogonal frame for g and $\hat{g}$. Moreover, $\hat{g}(X_A, X_A) = e^{2\mu_A}$.

Assumptions, bounds

Weighted Sobolev norms: Assume

$$\|\mathcal{K}(\cdot,t)\|_{H^l_{\mathfrak{u}}(\bar{M})} := \left(\int_{\bar{M}} \sum_{m=0}^{l} \langle \varrho(\cdot,t)\rangle^{-2m\mathfrak{u}} |\bar{D}^m \mathcal{K}(\cdot,t)|^2_{\bar{g}_{\mathrm{ref}}} \mu_{\bar{g}_{\mathrm{ref}}}\right)^{1/2}$$

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to be bounded and similarly for $\hat{\mathcal{L}}_U \mathcal{K}$.

Off diagonal components of $\hat{\mathcal{L}}_U \mathcal{K}$. $\{Y^A\}$ dual to $\{X_A\}$. Assume $(\hat{\mathcal{L}}_U \mathcal{K})(Y^A, X_B)$ to decay exponentially for $B > 1$ and $A \neq B$.

Assumptions, continued

Mean curvature. Assume $\bar{D} \ln \theta$ and q to be bounded in weighted Sobolev spaces. Here q is the *deceleration parameter*:

$$\hat{U}(n \ln \theta) = -1 - q.$$

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Lapse and shift. $\partial_t = NU + \chi$ defines the *shift vector field* χ and the *lapse function* N.

Equations

System of linear wave equations:

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Reformulation:

$$-\hat{U}^2 u + \sum_A e^{-2\mu_A} X_A^2 u + Z^0 \hat{U} u + Z^A X_A u + \hat{\alpha} u = 0,$$

$$Z^0 := \frac{1}{n}[q - (n-1)]\mathrm{Id} + \hat{\mathcal{X}}^0.$$

Results

Time coordinate and energies. Let $\tau(t) := \varrho(\bar{x}_0, t)$. Then

$$\hat{E}[u](t) := \frac{1}{2} \int_{\bar{M}_t} \left(|\hat{U}(u)|^2 + \sum_A e^{-2\mu_A} |X_A(u)|^2 + |u|^2 \right) \theta \mu_{\bar{g}}$$

and higher order versions thereof grow exponentially *at a rate independent of the order*.

Localising the equations

Localisation: γ inextendible causal curve. Spatial component converges to $\bar{x}_0$. Then

- In $J^+(\gamma)$ the time τ and ϱ are comparable.
- $\hat{U}(\varrho) \approx 1 \rightsquigarrow \hat{U} \approx \partial_\tau$ in $J^+(\gamma)$.
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Model equation: Replace $\hat{U}$ with ∂_τ; omit spatial derivatives; and localise the remaining coefficients:

$$-v_{\tau\tau} + Z^0_{\mathrm{loc}} v_\tau + \hat{\alpha}_{\mathrm{loc}} v = 0.$$

Here $Z^0_{\mathrm{loc}}(t) := Z^0(\bar{x}_0, t)$ and $\hat{\alpha}_{\mathrm{loc}}(t) := \hat{\alpha}(\bar{x}_0, t)$.

Results

Model equation. The model equation takes the form $\Psi_\tau = A\Psi$, where

$$\Psi := \begin{pmatrix} v \\ v_\tau \end{pmatrix}, \quad A := \begin{pmatrix} 0 & \mathrm{Id} \\ \hat{\alpha}_{\mathrm{loc}} & Z^0_{\mathrm{loc}} \end{pmatrix}.$$

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$$\|\Phi(s_1;s_2)\| \leq C_A\langle s_2 - s_1\rangle^{d_A} e^{\varpi_A(s_1-s_2)}.$$

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$$|\hat{U}u| + |u| \leq C\langle\varrho\rangle^{d_A} e^{\varpi_A \varrho}$$

in $J^+(\gamma)$.

Asymptotics

Assumptions. Assume Z^0_{loc} and $\hat{\alpha}_{\mathrm{loc}}$ to converge exponentially to Z^0_∞ and $\hat{\alpha}_\infty$ respectively. Let

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Conclusions, asymptotics. If u is a solution, there is a $\beta > 0$ and a V_∞ such that

$$\left| \begin{pmatrix} u \\ \hat{U} u \end{pmatrix} - e^{A_0 \varrho} V_\infty \right| \leq C e^{(\varpi_A + \beta)\varrho}$$

in $J^+(\gamma)$.

Thank you!